(1.8g) Range(f), where $f(x) = e^{-\frac{x+1}{x}}$, $f:(0,\infty)\to\mathbb{R}$

$$\text{Range}(f) = \{ f(x) : x \in \text{domain of } f \}.$$

$$= \{ e^{-\frac{x+1}{x}} : x > 0 \}.$$

You may fact that $G(x) = e^x$ is an strictly increasing fcn., so e.g. 'if $A \leq B$, then $g(A) \leq g(B)$.

Limits.

A sequence $(a_n) = (a_1, a_2, a_3, \dots)$

$$= (a_n)_{n=1}^{\infty} \quad \dots$$

input

is a function from $\mathbb{N}$ to $\mathbb{R}$.

(ie. $f: \mathbb{N} \to \mathbb{R}$, $f(k) = a_k$.)

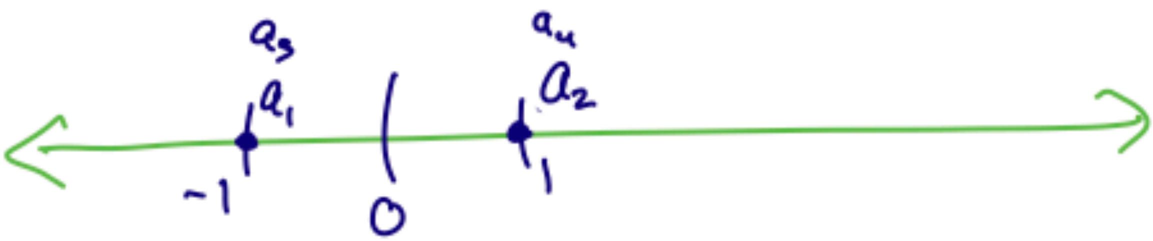
Examples

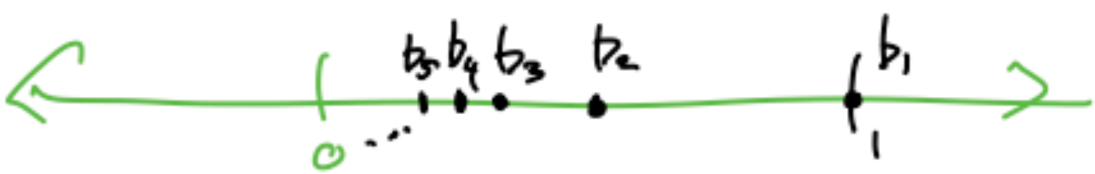
$$a_n = (-1)^n$$

$$b_n = \frac{1}{n}$$

$$c_n = \sqrt{n}$$

Picture on number line



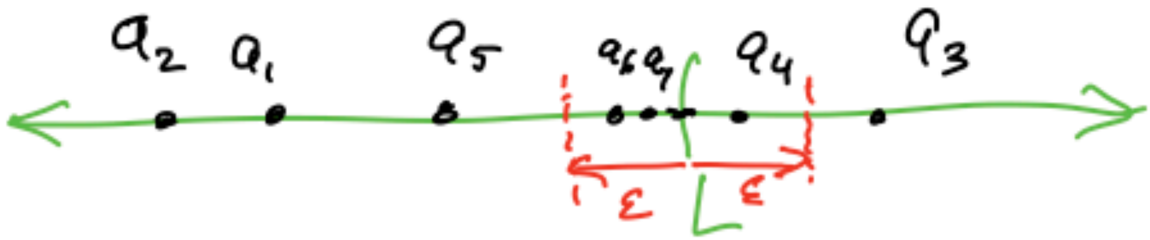


Defn. We say that $\lim a_n = L$ (or $a_n \to L$, "a_n converges to L") if, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$$\forall n \geq N \quad |a_n - L| < \varepsilon.$$

(distance between a_n & L) $< \varepsilon$

picture of the definition of limit.



$N = 6$, $a_6, a_7, a_8, \dots$ are inside the ε-neighborhood of L.

Example Prove that the sequence defined by $x_n = \dfrac{2n-3}{n+5}$ converges.

Scratch work: limit $= L = 2$.

Want: $|a_n - L| < \varepsilon$

$$\Leftrightarrow \left| \frac{2n-3}{n+5} - 2 \right| < \varepsilon$$

"solving for n". $\vdots$ $\vdots$

$$\Leftarrow n \geq \text{something}$$

usually depends on ε.

use to pick N

$$\left|\frac{2n-3}{n+5}-2\right|<\varepsilon$$

$$\Updownarrow$$

$$\left|\frac{2n-3}{n+5}-\frac{2(n+5)}{n+5}\right|<\varepsilon$$

$$\Updownarrow$$

$$\left|\frac{2n-3-2n-10}{n+5}\right|<\varepsilon$$

$$\Updownarrow$$

$$\left|\frac{-13}{n+5}\right|<\varepsilon$$

neg.

positive

$$\Updownarrow$$

$$\frac{13}{n+5}<\varepsilon$$

$$\Leftrightarrow \quad 13<\varepsilon(n+5)$$

$$\Leftrightarrow \quad 13<\varepsilon n+5\varepsilon$$

$$\Leftrightarrow \quad 13-5\varepsilon<\varepsilon n$$

$$\Leftrightarrow \quad \frac{13}{\varepsilon}-5<n$$

$$\Updownarrow$$

$$n>\frac{13}{\varepsilon}-5$$

let $N=\left\lceil\frac{13}{\varepsilon}-5\right\rceil+5$ ceiling

$\lceil 3.8\rceil=4$

$\lceil 2.1\rceil=3.$

Actual Proof: $\forall \varepsilon > 0$, Let

$$N = \left\lceil \frac{13}{\varepsilon} - 5 \right\rceil + 5 \in \mathbb{N}.$$

Then, if $n \geq N$,

$$\left| \frac{2n-3}{n+5} - 2 \right|$$

$$= \left| \frac{-13}{n+5} \right| = \frac{13}{n+5} \leq \frac{13}{\left(\left\lceil \frac{13}{\varepsilon} - 5 \right\rceil + 5 \right) + 5}$$

$$= \frac{13}{\left\lceil \frac{13}{\varepsilon} - 5 \right\rceil + 10} \leq \frac{13}{\frac{13}{\varepsilon} - 5 + 10} = \frac{13}{\frac{13}{\varepsilon} + 5}$$

$$< \frac{13}{\frac{13}{\varepsilon}} = \varepsilon.$$

$\therefore \lim \frac{2n-3}{n+5} = 2.$ $\blacksquare$

<u>Example</u> Find $\lim \frac{1}{2\sqrt{n}+1}$ and prove it exists.

<u>Scratchwork.</u>

Want $\left|\frac{1}{2\sqrt{n}+1} - 0\right| < \varepsilon.$

$$\Leftrightarrow \frac{1}{2\sqrt{n}+1} < \varepsilon$$

$$\Leftrightarrow \frac{1}{\varepsilon} < 2\sqrt{n}+1$$

$$\Leftrightarrow \frac{1}{\varepsilon} - 1 < 2\sqrt{n}$$

$$\Leftrightarrow \frac{1}{2\varepsilon} - \frac{1}{2} < \sqrt{n}$$

$$-\frac{1}{2} \leq \frac{1}{2\varepsilon} - \frac{1}{2} \qquad \sqrt{n} \geq 1$$

$$\left(\frac{1}{2\varepsilon} - \frac{1}{2}\right)^2 < n$$

OK this time

Let $N = \left\lceil \left(\frac{1}{2\varepsilon} - \frac{1}{2}\right)^2 \right\rceil + 1$

Actual Proof $\forall \varepsilon > 0$, let

$$N = \left\lceil \left(\frac{1}{2\varepsilon} - \frac{1}{2}\right)^2 \right\rceil + 1. \text{ If } n \geq N,$$

Then

$$\left| \frac{1}{2\sqrt{n}+1} - 0 \right| = \frac{1}{2\sqrt{n}+1}$$

$$\leq \frac{1}{2\sqrt{\left\lceil \left(\frac{1}{2\varepsilon} - \frac{1}{2}\right)^2 \right\rceil + 1} + 1}$$

$$< \frac{1}{2\sqrt{\left(\frac{1}{2\varepsilon} - \frac{1}{2}\right)^2} + 1} = \frac{1}{2\left|\frac{1}{2\varepsilon} - \frac{1}{2}\right| + 1}$$

$$\left(\sqrt{x^2} = |x|\right)$$

$$\leq \frac{1}{2\left(\frac{1}{2\varepsilon} - \frac{1}{2}\right) + 1},$$

since $-1 < 2\left(\frac{1}{2\varepsilon} - \frac{1}{2}\right) < 2\left|\frac{1}{2\varepsilon} - \frac{1}{2}\right|$.

& then $0 < 2\left(\frac{1}{2\varepsilon} - \frac{1}{2}\right) + 1 < 2\left|\frac{1}{2\varepsilon} - \frac{1}{2}\right| + 1$.

Then

$$\left| \frac{1}{2\sqrt{n}+1} - 0 \right| < \frac{1}{\frac{1}{\varepsilon} - 1 + 1} = \frac{1}{\frac{1}{\varepsilon}}$$

$$= \varepsilon.$$

Thus, $\lim \frac{1}{2\sqrt{n}+1} = 0$. $\boxtimes$

Proving a limit does not exist:

The limit $\lim a_n$ exists

$\iff \exists L \in \mathbb{R}$ s.t. $\forall \varepsilon > 0$
$\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $|a_n - L| < \varepsilon$.

The limit $\lim a_n$ does not exist.

$\iff \neg \big(\exists L \in \mathbb{R}$ s.t. $\forall \varepsilon > 0$
$\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $|a_n - L| < \varepsilon. \big)$

$\iff \forall L \in \mathbb{R}, \exists \varepsilon > 0$ s.t.
$\forall N \in \mathbb{N}, \exists n \geq N$ s.t. $|a_n - L| \geq \varepsilon$.